(3 Hours) [Total Marks : 100

N.B. : (1) Question No. **1** is **compulsory**.
(2) Answer any **four** out of the remaining **six** questions.
(3) **Figures** to the **right** indicate **full** marks.

1. (a) State and prove the change of scale property of Laplace transforms. If $L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-(1/4s)}$, find $L(\sin 2\sqrt{t})$. **5**

(b) Show that every square matrix can be uniquely expressed as P + iQ where P and Q are Hermitian matrices. **5**

(c) Find the Z transform of $\frac{1}{k+1}$, $k \geq 0$. Indicate the region of convergence. **5**

(d) Find the Fourier series expansion of $f(x) = x - x^2$, $-1 < x < 1$. **5**

2. (a) Find non-singular matrices P and Q such that the normal form of **8**

$$A = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix}$$ is PAQ. What is the rank of A ?

(b) Show that the set of functions $\sin(2n + 1)x$, $n = 0, 1, 2, \ldots\ldots$ is orthogonal over $\left[0, \frac{\pi}{2}\right]$. Hence construct an orthonormal set of functions. **6**

(c) Obtain $L(\text{erf}\sqrt{t})$. Hence evaluate $\int_0^{\infty} te^{-t^2}\,\text{erf}(t)\,dt$. **6**

3. (a) Obtain (i) $L\left((1+te^{-t})^3\right)$ (ii) $L\left(\frac{\cosh 2t \sin 2t}{t}\right)$ **8**

(b) Find the inverse Z-transform of $\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}$, for **6**

(i) $|z|>1$, (ii) $|z|<1/2$, (iii) $1/2<|z|<1$.

(c) Find l, m, n and A^{-1} if $A = \begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}$ is orthogonal. **6**

4. (a) (i) Evaluate $\int_0^{\infty} e^{-2t} \sin^3 t\,dt$ using Laplace transforms. **8**

(ii) If $\int_0^{\infty} e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{3}{8}$, find α.

(b) Find the Fourier integral representation of the function – **6**

$$f(x) = \begin{cases} e^{ax}, & x \leq 0 \\ e^{-ax}, & x \geq 0 \end{cases} \text{ for } a > 0.$$

(c) Obtain the Fourier series of $f(x) = \sqrt{1-\cos x}$ in the interval $(0, 2\pi)$. 6

Deduce that $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.

5. (a) Find (i) $L^{-1}\left(e^{-s}\left(\frac{(1+\sqrt{s})}{s^3}\right)\right)$ 8

(ii) $L^{-1}\left(\frac{(s+2)^2}{(s^2+4s+8)^2}\right)$

(b) Solve by the Gauss elimination method – 6

$$\begin{aligned} 2x + 5y + 2z - 3w &= 3 \\ 3x + 6y + 5z + 2w &= 2 \\ 4x + 5y + 14z + 14w &= 11 \\ 5x + 10y + 8z + 4w &= 4 \end{aligned}$$

(c) Find the half-range cosine series of $f(x) = x(\pi - x)$, in the interval $[0, \pi]$. 6

Deduce that (i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$

6. (a) Discuss the values of k for which the following system of equations – 8

$$\begin{aligned} 2x + 3ky + (3k+4)z &= 0 \\ x + (k+4)y + (4k+2)z &= 0 \\ x + 2(k+1)y + (3k+4)z &= 0 \end{aligned}$$

has non trivial solutions. Also find the solutions.

(b) Solve using Laplace transforms : $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t,\ y(0) = 1$ 6

(c) Find the Fourier series of $f(x) = x\,|x|$ in $(-1,1)$ 6

7. (a) Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$. Hence evaluate 8

$\int_0^{\infty} \tan^{-1}\left(\frac{x}{a}\right) \sin x\,dx$.

(b) State the convolution theorem for z-transform. Use the theorem to find z(h(k)) 6
where h(k) is the convolution of $f_1(k) = \frac{1}{3^k}, k \geq 0$ and $f_2(k) = \frac{1}{4^k}, k \geq 0$.

(c) Find the complex form of the Fourier series for $f(x) = 2x$ in $(0, 2\pi)$. 6