F.E. (Sem-I) \ maths-I \ Dec 09

Con. 3680-09. **SP–8453**

(3 Hours) **[Total Marks : 100**

N.B. : (1) Question No. **1** is **compulsory**.
(2) Attempt any **four** questions out of remaining **six** questions.

1. (a) If $x + iy = \sqrt[3]{a + ib}$, prove that $\frac{a}{x} + \frac{b}{y} = 4\left(x^2 - y^2\right)$. 5

(b) If $u = \log(\tan x + \tan y + \tan z)$, 5

prove that $(\sin 2x)\frac{\partial u}{\partial x} + (\sin 2y)\frac{\partial u}{\partial y} + (\sin 2z)\frac{\partial u}{\partial z} = 2$

(c) If $\bar{l}, \bar{m}, \bar{n}$ are three non-coplanar vectors, prove that – 5

$$\left[\bar{l}\ \bar{m}\ \bar{n}\right]\left(\bar{a} \times \bar{b}\right) = \begin{vmatrix} \bar{l}\cdot\bar{a} & \bar{l}\cdot\bar{b} & \bar{l} \\ \bar{m}\cdot\bar{a} & \bar{m}\cdot\bar{b} & \bar{m} \\ \bar{n}\cdot\bar{a} & \bar{n}\cdot\bar{b} & \bar{n} \end{vmatrix}$$

(d) Test for convergence of the series $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots\ldots\ldots (p > 0)$ 5

2. (a) If $\sin^4\theta\cdot\cos^3\theta = a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta + a_7\cos 7\theta$ 6
Prove that $a_1 + 9a_3 + 25a_5 + 49a_7 = 0$

(b) If $y = 2^x \sin^2 x\cdot\cos^3 x$, find y_n. 7

(c) Find the value of a and b such that $\lim_{x\to 0} \frac{a\sin^2 x + b\log\cos x}{x^4} = \frac{1}{2}$ 7

3. (a) Prove that $\left(\frac{1 + \tanh x}{1 - \tanh x}\right)^3 = \cosh 6x + \sinh 6x$ 6

(b) If $u = f\left(e^{y-z}, e^{z-x}, e^{x-y}\right)$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ 7

(c) Prove that $\nabla\cdot\left(r\,\nabla\frac{1}{r^n}\right) = \frac{n(n-2)}{r^{n+1}}$. 7

4. (a) Prove that $(1 + i\tan\alpha)^{-i} = e^{(2m\pi + \alpha)}\left[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)\right]$ 6

(b) Verify Euler's theorem for $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ 7

and also prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{x^2 - y^2}{x^2 + y^2}$

(c) Find f(r) so that the vector $f(r)\,\bar{r}$ is both solenoidal and irrotational. 7

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5. (a) If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, prove that $(1-x^2)y_{n+1} - (2n+1)xy_n - n^2 y_{n-1} = 0$ 6

(b) If $\sin\alpha + \sin\beta + \sin\gamma = 0$ and $\cos\alpha + \cos\beta + \cos\gamma = 0$, prove that 7

(i) $\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = 0$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

(c) State Rolle's theorem and use it to find 'c' for $\log\left[\frac{x^2+ab}{(a+b)x}\right]$ in [a, b], a, b > 0. 7

6. (a) Show that minimum value of $u = xy + a^3\left(\frac{1}{x}+\frac{1}{y}\right)$ is $3a^2$. 6

(b) Find the roots common to $x^4 + 1 = 0$ and $x^6 - i = 0$. 7

(c) If $u = \operatorname{cosec}^{-1}\sqrt{\left[\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right]}$, prove that 7

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left[\frac{13}{12}+\frac{\tan^2 u}{12}\right].$$

7. (a) Find the approximate value of 6

$$\left[(0{\cdot}98)^2 + (2{\cdot}01)^2 + (1{\cdot}94)^2\right]^{1/2}$$

(b) Prove that (i) $\tan h^{-1}x = \sin h^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ 7

(ii) $\sin h^{-1}x = \cos h^{-1}\left(\sqrt{1+x^2}\right)$

(c) If $u = \frac{x^2y^2z^2}{x^2+y^2+z^2} + \cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$, 7

find the value of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$